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Analytic properties of the Constantin-Lax-Majda equation with a generalized viscosity term

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My memory with Professor Richard Pelz

It was summer in 2002 when I met Professor Pelz for the first time. He had been staying in RIMS at Kyoto University as a visiting professor at that time. I gave a talk at a seminar there and he was in the audience. After then, I visited him. He was interested in the topic that I talked at the seminar and made insightful comments about it. I also talked with him about the APS/DFD conference that was to be held at Dallas that year. I told him that I was not able to make a presentation about my recent research at the conference though I hoped to do so, since I was not a member of the APS. Then he willingly offered to introduce me to the APS, which made me possible to participate in the conference as a speaker.

This article contains a survey of the research that I could talk there thanks to his kindness. I would say the research is never completed without his introduction. I am very grateful to him for his kindness and generosity. I dedicate this article to my short but wonderful memories with him.

1 Introduction

We consider Euler equation for inviscid and incompressible fluids in $\mathbf{R}^3$ space:

$$\left\{\begin{array}{lcll} \dfrac{Dw}{Dt} & = & w\nabla v, & x \in \mathbf{R}^3,\ t>0, \\ w(x,0) & = & w_0(x), & \end{array}\right. \tag{1}$$

where $\frac{D}{Dt}$ is the convective derivative, v is the velocity field that satisfies div $v=0$ and $w=\nabla\times v$ is the vorticity. The velocity and the vorticity appear in the equation, but this is in fact only for the vorticity, since the Biot-Savart integral recovers the velocity from the vorticity.

$$v(x,t) = -\frac{1}{4\pi}\int \frac{(x-y)}{|x-y|^3} \times w(y,t)dy. \tag{2}$$

It is quite difficult to show well-posedness of the equations in three-dimensional space, since the quadratic term $w\nabla v \neq 0$ allows the vorticity to grow largely. According to Beale et al.[1], the maximum vorticity necessarily diverges in finite time, if the smooth solution loses its analyticity. Hence, the growth of the vorticity due to the quadratic term is a key factor for the blow-up.

One of the trials for the mathematical problem is to construct a model equation to observe the analytic property of the quadratic term. Constantin, Lax and Majda[2] introduced a one-dimensional model for the Euler equations.

$$\frac{\partial \omega}{\partial t} = \omega H(\omega), \qquad x \in \mathbf{R},\ t > 0.$$

This is called Constantin-Lax-Majda equation. The operator H is the Hilbert transform and $\omega H(\omega)$ is a scalar one-dimensional analogue of the quadratic term. They gave an explicit solution that blows up in finite time and discuss the similarity between the Euler equations and the model equation. See [2] for the details.

Another important topic in the field of mathematical fluid dynamics is well-posedness of the Navier-Stokes equation in three-dimensional space. Now, we consider the CLM equation with the diffusion term, which gives a heuristic model for the 3-D Navier-Stokes equations. The simplest model was considered by Schochet[7].

$$\frac{\partial \omega}{\partial t} = \omega H(\omega) + \nu \omega_{xx}, \qquad x \in \mathbf{R},\ t > 0.$$

He found a solution that blowed up in finite time and showed that the velocity recovered from the solution also blowed up at the same time. He also concluded that the model with the simple diffusion term was less successful than the CLM equation, since the solution of the model equation blowed up before the solution of the original CLM equation.

On the other hand, Murthy[3] and Wegert & Murthy[8] chose an non-local viscosity term, $\nu H(\omega_x)$ as the diffusion term. They gave a solution that blowed up after the solution of the CLM equation blowed up.

Now, we consider the CLM equation with a generalized viscosity term:

$$\begin{cases} \dfrac{\partial \omega}{\partial t} &= \omega \mathrm{H}(\omega) - \nu(-\Delta)^{\frac{\alpha}{2}}\omega, \qquad x \in \mathbf{R},\ t > 0, \\ \omega(x,0) &= \omega_0(x), \end{cases} \tag{3}$$

where a positive real ν represents the viscosity coefficient and $\alpha \in \mathbf{R}$ is order of derivative. Well-posedness of the generalized CLM equation is studied by S-, recently[5] and [6]. In the present paper, we outline the results in these papers and add some new materials in terms of the singularity formation.

This paper contains four sections. We give a spectral representation of the CLM equation (3) and give a formal solution in §2. In §3, we show blow-up and global existence of solutions when initially all Fourier coefficients are non-negative. In §4, we show the distribution of singularity times in the complex-time domain to see the analytic property clearly.

2 Spectral Method

We assume that the periodic boundary condition in x; $\omega(x+2\pi,t) = \omega(x,t)$. Thus the following Fourier series represents a formal solution of the generalized CLM equation;

$$\omega(x,t) = \sum_{n=-\infty}^{\infty} \omega_n(t) \mathrm{e}^{\mathrm{i}nx}, \quad \omega_n(t) \in \mathbf{C}. \tag{4}$$

The Hilbert transform of (4) is expressed as follows[4].

$$\mathrm{H}(\omega) = \sum_{n=-\infty}^{\infty} \mathrm{i}\ \mathrm{sgn}(n)\omega_n(t)\mathrm{e}^{\mathrm{i}nx},$$

where the function sgn(n) is

$$\mathrm{sgn}(n) = \begin{cases} 1 & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -1 & \text{if } n < 0. \end{cases}$$

Then, the quadratic term $\omega \mathrm{H}(\omega)$ was simply given as follows[5].

$$\omega \mathrm{H}(\omega) = \sum_{n=1}^{\infty} \left\{ \left(\mathrm{i} \sum_{k=0}^{n} \omega_k \omega_{n-k} \right) \mathrm{e}^{\mathrm{i}nx} - \left(\mathrm{i} \sum_{k=0}^{n} \omega_{-k} \omega_{-n+k} \right) \mathrm{e}^{-\mathrm{i}nx} \right\}. \quad (5)$$

Substituting (4) and (5) into (3) and equating coefficients of $\mathrm{e}^{\mathrm{i}nx}$, we obtain infinite dimensional ordinary differential equations for the spectra;

$$\begin{cases} \dfrac{\mathrm{d}\omega_n}{\mathrm{d}t} = -\nu n^{\alpha} \omega_n + \mathrm{i} \displaystyle\sum_{k=0}^{n} \omega_k \omega_{n-k}, \\ \dfrac{\mathrm{d}\omega_0}{\mathrm{d}t} = -\nu n^{\alpha} \omega_0, \\ \dfrac{\mathrm{d}\omega_{-n}}{\mathrm{d}t} = -\nu n^{\alpha} \omega_{-n} - \mathrm{i} \displaystyle\sum_{k=0}^{n} \omega_{-k} \omega_{-n+k}. \end{cases} \quad (6)$$

We assume that initial condition of ω is symmetric with respect to the origin.

$$\omega(x, 0) = \sum_{n=0}^{\infty} A_n \sin nx. \quad (7)$$

Then, it follows from the equations (6) that the solution is symmetric with respect to the origin for all time. That is to say,

$$\omega_n(t) + \omega_{-n}(t) = 0, \qquad \text{for} \qquad t > 0.$$

Because of the symmetry, the function $\mathrm{i}\omega_n(t)$ becomes real-valued and thus the equations (6) and (7) are reduced to

$$\frac{\mathrm{d}p_n^{(\alpha)}}{\mathrm{d}t} = -\nu n^{\alpha} p_n^{(\alpha)} + \sum_{k=1}^{n-1} p_k^{(\alpha)} p_{n-k}^{(\alpha)}, \quad (8)$$

$$p_n^{(\alpha)}(0) = \frac{A_n}{2}, \qquad n = 1, 2, \cdots, \quad (9)$$

where $p_n^{(\alpha)}(t) \equiv \mathrm{i}\omega_n(t)$.

Let $p_n^{(\alpha)}(t; A_n)$ denotes the solutions of (8) with the initial conditions (9), which are given recursively by

$$p_1^{(\alpha)}(t; A_1) = \frac{A_1}{2} \mathrm{e}^{-\nu t}, \quad (10)$$

$$p_n^{(\alpha)}(t; A_n) = \frac{A_n}{2} \mathrm{e}^{-n^{\alpha} \nu t} + \mathrm{e}^{-n^{\alpha} \nu t} \int_0^t \mathrm{e}^{n^{\alpha} \nu s} \sum_{k=1}^{n-1} p_k^{(\alpha)} p_{n-k}^{(\alpha)} ds. \quad (11)$$

Note that the solution of the equation (3) for the initial condition (7) is represented formally by

$$\omega(x, t) = \sum_{n=1}^{\infty} 2 p_n^{(\alpha)}(t; A_n) \sin nx.$$

3 Blow-up and global existence of solutions

In this section, we outline the blow-up and global results for the generalized CLM equation. We assume that the initial spectra A_n are non-negative. As for the detailed proof, please refer to the papers [5] and [6].

3.1 Two comparisons

The following two comparisons are required to show blow-up and global existence of solutions for general non-negative initial spectra and arbitrary α from a solution for special initial data. The first comparison is obtained in terms of the index α.

Proposition 1 *If $\alpha_2 \geq \alpha_1$, then $p_n^{(\alpha_1)}(t; A_n) \geq p_n^{(\alpha_2)}(t; A_n) \geq 0$ for $n \geq 1$ and $t \geq 0$.*

The second comparison is about the initial conditions.

Proposition 2 *If $A_n \geq \tilde{A}_n \geq 0$, then $p_n^{(\alpha)}(t; A_n) \geq p_n^{(\alpha)}(t; \tilde{A}_n) \geq 0$ for $n \geq 1$ and $t \geq 0$.*

3.2 Blow-up solutions

3.2.1 Explicit solution for $\alpha = 1$

When $\alpha = 1$, we are able to give an explicit expression of $p_n^{(\alpha)}(t; A_1\delta_{1n})$, where $A_1 > 0$ and δ_{1n} is Kronecker's delta.

Lemma 3 *When $\alpha = 1$, the solutions (10) and (11) for $A_n = A_1\delta_{1n}$ are represented by*

$$p_n^{(\alpha)}(t; A_1\delta_{1n}) = \left(\frac{A_1}{2}\right)^n t^{n-1} \mathrm{e}^{-n\nu t}. \tag{12}$$

This lemma is shown easily by the mathematical induction. Then, this lemma and Proposition 2 indicate the blow-up solutions for the general non-negative initial data, when the viscosity coefficient is sufficiently small.

Theorem 4 *When $\alpha = 1$, for $0 < \nu \leq \frac{A_1}{2\mathrm{e}}$, there exists a finite time $T_1^*(\nu)$ such that $||\omega(x,t)||_{L^2} \to \infty$ as $t \to T_1^*(\nu)$.*

3.2.2 Blow-up solutions for $\alpha > 1$

For $\alpha > 1$, we obtain the following blow-up solutions for non-negative initial spectra.

Theorem 5 *When $\alpha > 1$, for $0 < \nu \leq \frac{A_1}{2}\mathrm{e}^{-3\alpha-1}$, there exists a finite time $T_2^*(\nu)$ such that $||\omega(x,t)||_{L^2} \to \infty$ as $t \to T_2^*(\nu)$.*

The first step to prove the theorem is to define a sequence $\left\{b_n^{(\alpha)}\right\}_{n\geq 1}$ recursively by

$$b_1^{(\alpha)} = 1, \qquad b_n^{(\alpha)} = \frac{1}{n^\alpha}\sum_{k=1}^{n-1} b_k^{(\alpha)} b_{n-k}^{(\alpha)} \quad \text{for } n = 2, 3, \cdots.$$

We give a lower bound for $b_n^{(\alpha)}$.

Lemma 6 *For $\alpha > 1$, let D_α be a constant such that $\alpha 2^{3\alpha} < D_\alpha \le e^{3\alpha}$. Then, the sequence $b_n^{(\alpha)}$ is bounded as follows:*

$$e^{-3\alpha n} < D_\alpha n^{\alpha-1} e^{-3\alpha n} < b_n^{(\alpha)}. \tag{13}$$

Next lemma gives a lower estimate for the special solution $p_n^{(\alpha)}(t; A_1\delta_{1n})$.

Lemma 7 *Let functions $f_n^{(\alpha)}(t)$ be defined by*

$$p_n^{(\alpha)}(t) = \nu b_n^{(\alpha)} \left(\frac{A_1}{2\nu} e^{-\nu t}\right)^n f_n^{(\alpha)}(t). \tag{14}$$

Then, for $0 \le \nu t \le 1$, the functions $f_n^{(\alpha)}(t)$ satisfy

$$f_n^{(\alpha)}(t) \quad \ge \quad (\nu t)^{n-1}. \tag{15}$$

We finally prove Theorem 5 by using these lemma.

Proof of Theorem 5. From Parseval's equality and Proposition 2, we have

$$\begin{aligned}
||\omega(x,t)||^2_{L^2[-\pi,\pi)} &= \pi \sum_{n=1}^{\infty} \left\{2p_n^{(\alpha)}(t; A_n)\right\}^2 \\
&\ge \pi \sum_{n=1}^{\infty} \left\{2p_n^{(\alpha)}(t; A_1\delta_{1n})\right\}^2 = \pi \sum_{n=1}^{\infty} \left\{2\nu p_n^{(\alpha)} \left(\frac{A_1}{2\nu} e^{-\nu t}\right)^n f_n^{(\alpha)}(t)\right\}^2.
\end{aligned}$$

It follows from Lemma 6 and Lemma 7 that for $0 \le \nu t \le 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \left\{2\nu p_n^{(\alpha)} \left(\frac{A_1}{2\nu} e^{-\nu t}\right)^n f_n^{(\alpha)}(t)\right\}^2 &> \sum_{n=1}^{\infty} \left\{A_1 e^{-\nu t} e^{-3\alpha n} \left(\frac{A_1 t}{2} e^{-\nu t}\right)^{n-1}\right\}^2 \\
&= A_1^2 e^{-2\nu t} e^{-6\alpha} \sum_{n=1}^{\infty} \left(\frac{A_1 t}{2e^{3\alpha}} e^{-\nu t}\right)^{2(n-1)}.
\end{aligned}$$

Therefore, if $R(t) = \frac{A_1 t}{2e^{3\alpha}} e^{-\nu t}$ is less than one, we obtain

$$\begin{aligned}
||\omega(x,t)||^2_{L^2} &> \pi A_1^2 e^{-2\nu t} e^{-6\alpha} \sum_{n=1}^{\infty} \left\{R^2(t)\right\}^{n-1} \\
&= \pi \frac{A_1^2 e^{-2\nu t} e^{-6\alpha}}{1 - R^2(t)}.
\end{aligned} \tag{16}$$

Since $R(t)$ is monotonically increasing for $0 \le \nu t \le 1$, there exists a time $T_2^*(\nu) \in [0, \frac{1}{\nu}]$ such that $R\left(T_2^*(\nu)\right) = 1$, if the viscosity coefficient satisfies

$$R\left(\frac{1}{\nu}\right) = \frac{A_1}{2\nu e^{3\alpha+1}} \ge 1.$$

Hence, it follows from (16) that if $0 < \nu \le \frac{A_1}{2} e^{-3\alpha-1}$, $||\omega(x,t)||_{L^2}$ blows up as $t \to T_2^*(\nu)$ □

3.2.3 Blow-up solutions for $\alpha < 1$

We consider blow-up of the solution (10) and (11) when the order of viscosity term is less than one ($\alpha < 1$). We give a lower comparison function for the spectra $p_n^{(\alpha)}(t; A_1\delta_{1n})$.

Lemma 8 *For $\alpha < 1$, $n \geq 1$ and $t \geq 0$,*

$$p_n^{(\alpha)}(t; A_1\delta_{1n}) \geq \left(\frac{A_1}{2}\right)^n t^{n-1} \mathrm{e}^{-n\nu t}. \tag{17}$$

Therefore, we obtain the blow-up theorem for $\alpha < 1$ owing to Proposition 2 and Lemma 8.

Theorem 9 *When $\alpha < 1$, for $0 < \nu \leq \frac{A_1}{2e}$, there exists a finite time $T_1^*(\nu)$ such that $||\omega(x,t)||_{L^2} \to \infty$ as $t \to T_1^*(\nu)$.*

Consequently, the solution of the CLM equation with the generalized viscosity term for non-negative initial Fourier coefficients blows up in finite time regardless of the index α, if the viscosity coefficient is sufficiently small.

3.3 Global solutions

3.3.1 Global Existence of solutions for $\alpha \geq 1$

First of all, we consider local existence of solution of (3) for $\alpha \geq 1$ using the general theory of the abstract evolution equations. Since the linear operator $A_0 = \nu(-\Delta)^{\frac{\alpha}{2}}$ is the infinitesimal generator of a C^0 group on L^2, the initial value problem with the periodic boundary condition on $\Omega = [-\pi, \pi)$ can be expressed by

$$\left\{ \begin{array}{rcll} \dfrac{\partial \omega}{\partial t} + A_0\omega &=& F(\omega), & \text{in } L^2(\Omega),\ t > 0, \\ \omega(0) &=& \omega_0, & \end{array} \right. \tag{18}$$

where the nonlinear mapping $F(\omega) = \omega H(\omega)$. The nonlinear operator $F(\omega)$ is Lipschitz continuous in $H^1(\Omega)$.

Lemma 10 *The nonlinear operator $F(u)$ maps $H^1(\Omega)$ into $H^1(\Omega)$ and satisfies the following properties; for $u, v \in H^1(\Omega)$ there exist constants C and C' such that*

$$||F(u)||_{H^1} \leq C||u||_{H^1} \tag{19}$$
$$||F(u) - F(v)||_{H^1} \leq C'||u - v||_{H^1}. \tag{20}$$

Hence, it follows from Lemma 10 that the initial value problem (18) has a unique local solution for $\alpha \geq 1$.

Proposition 11 *For every $\omega_0 \in H^\alpha(\Omega) = D(A_0)$, there exist a time T_m and a unique solution of the initial value problem (18) such that*

$$\omega \in C^1([0, T_m) : L^2(\Omega)) \cap C([0, T_m) : H^\alpha(\Omega)).$$

Now suppose that $\omega(x, 0) \in H^\alpha$ and $M = ||\omega(x, 0)||_{H^\alpha}$. Then we have an explicit representation of (10) and (11) with $A_n = M$.

Lemma 12 *When $\alpha = 1$, the functions (10) and (11) for $A_n = M$ $(n \geq 1)$ are represented explicitly by*

$$p_n^{(1)}(t; M) = \left(\frac{M}{2}\right)\left(1 + \frac{M}{2}t\right)^{n-1} e^{-n\nu t}. \tag{21}$$

Consequently, we have the following lemma.

Lemma 13 *When $\alpha \geq 1$, if $\omega(x,0) \in H^\alpha(\Omega)$, then there exists a constant $C(\nu)$ such that $||\omega(x,t)||_{H^\alpha} < C$ for $\nu > \frac{M}{2}$.*

Finally, Lemma 10 and Lemma 13 yield the existence of a unique global solution.

Theorem 14 *Suppose that $\omega_0 \in H^\alpha(\Omega)$ and all of its initial Fourier coefficients are non-negative. Then the initial value problem (18) has a unique global solution,*

$$u \in C\left([0,\infty) : H^\alpha(\Omega)\right) \cap C^1\left([0,\infty) : L^2(\Omega)\right),$$

for $\alpha \geq 1$ and $\nu > \frac{1}{2}||\omega_0||_{H^\alpha}$.

3.3.2 No global solution for $\alpha \leq 0$

For arbitrary ν, there exists a large integer N_0 such that the dissipation term $\nu n^\alpha p_n^{(\alpha)}$ in (8) becomes quite small for $n \geq N_0$, since $\alpha < 0$. Hence, we expect that the dissipation term fails to control the growth of the high spectra, which leads to the blow-up of the solution. The following lemma supports this expectation.

Lemma 15 *Let $\alpha < 0$. For arbitrary ν and $\epsilon > 0$, there exists a positive integer N_0 such that the solution of (3) for the initial condition $\omega(x,0) = 2\epsilon \sin N_0 x$ blows up in finite time in L^2 norm.*

The comparison in terms of the initial conditions (Proposition 2) and this lemma show that for given non-negative initial conditions $A_n \geq 0$, the viscosity coefficient ν and an arbitrary ϵ, there exists a positive integer N_0 such that the solution of (3) for the slightly perturbed initial condition $A_n + \epsilon\delta_{N_0 n}$ blows up in finite time. Since ϵ is arbitrary, we show that no global solution exists for $\alpha < 0$.

Theorem 16 *When $\alpha < 0$, the solution of (3) for initial condition with non-negative Fourier coefficients $A_n \geq 0$ always blows up in finite time in L^2 norm.*

Hence, the solution of (3) blows up in finite time no matter how large the viscosity coefficient is.

4 Singularity time in complex time domain

The results for $\alpha \geq 1$ in the previous section indicate that the solution of the CLM equation blows up in finite time when the viscosity coefficient is small, while it exists globally for large viscosity. It means the singularity abruptly disappears as the viscosity increases. In order to study the discontinuity in terms of the singularity formation, we investigate distribution of singular times in the complex time plane.

Now, assuming that the time variable is complex number, we regard the spectra $p_n^{(\alpha)}$ as a complex-valued function. First, choosing the integration path starting form the origin in the complex time plane, we compute (10) and (11) numerically along

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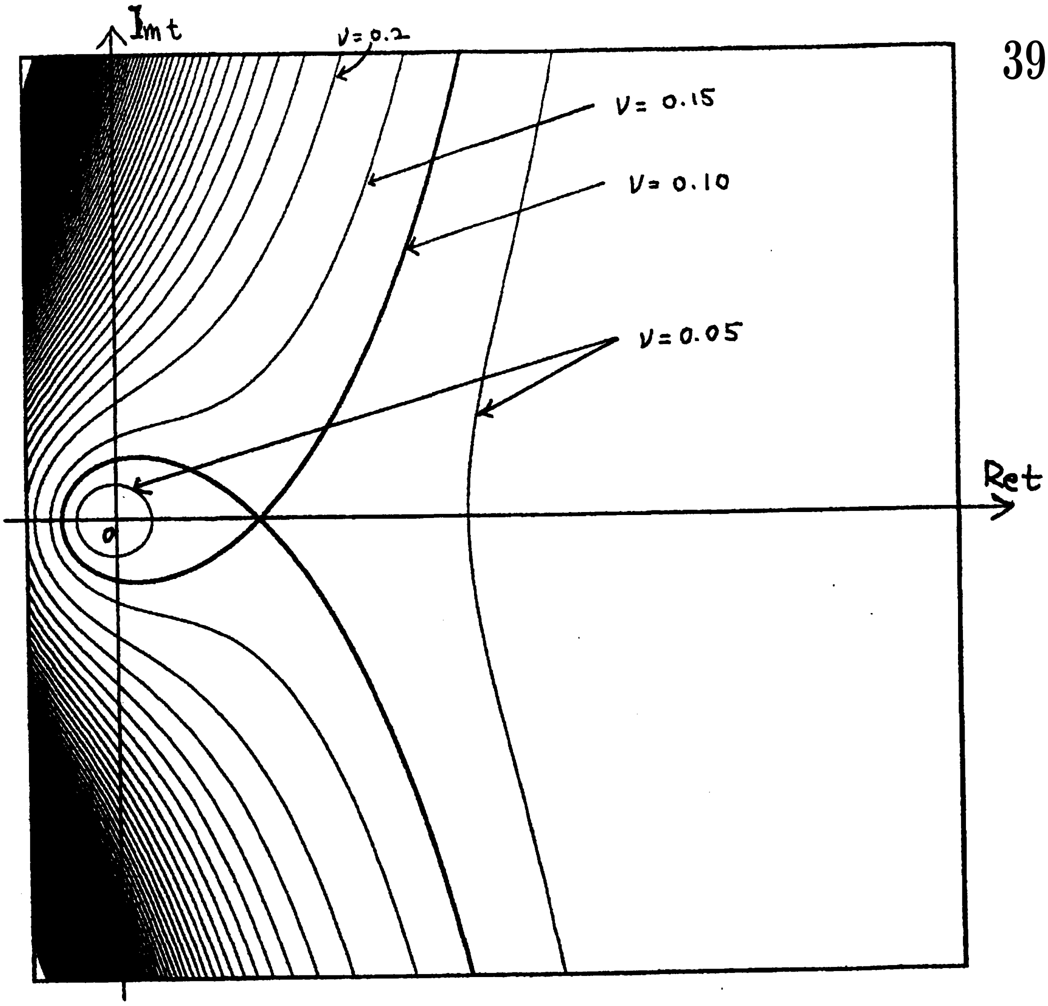

Figure 1: Distribution of singularity time in the complex-time plane for $\alpha = 1$

the path. Next, since the sufficient and necessary condition for the convergence of L^2 norm of the solution is

$$\lim_{n\to\infty}\left|\frac{p_{n+1}^{(\alpha)}}{p_n^{(\alpha)}}\right| = g(t) < 1,$$

we compute the critical time $t^* \in \mathbf{C}$ such that $g(t^*) = 1$ for various ν and plot them in the complex time domain. In the present paper, we show the distribution of the critical times for a specific initial condition, $A_n = \delta_{1n}$.

When $\alpha = 1$, we have the explicit representation of $p_n^{(\alpha)}$ in Lemma 3 and thus $g(t) = |\frac{t}{2}e^{-\nu t}|$. We plot the contour plot of $g(t)$ for various ν in Figure 1. For a small ν, the distribution of the singularity times consists of a circular line surrounding the origin and a vertical line in the right hand side of the figure. A region inside the circular distribution and an unbounded region whose boundary is the vertical line are regular regions in which the solution of the generalized CLM equation exists. The distribution has three intersections with the real time axis. Hence, when the path goes from the origin in the positive real-time direction, it meets one of the singularity times in the circular distribution at which the solution blows up. As the viscosity coefficient increases, the two disjoint regular regions unite and move to the left. Then, since the origin is always in the right hand side of the distribution, the solution along the path from the origin in the positive real-time direction exists globally. We must note that the solution always blows up along the negative real-time axis, since the path meets the distribution.

Next we show the distribution of the singularity times for $\alpha = 1.5$. In this case, since we have no explicit expression for the solution, we compute the function $g(t)$

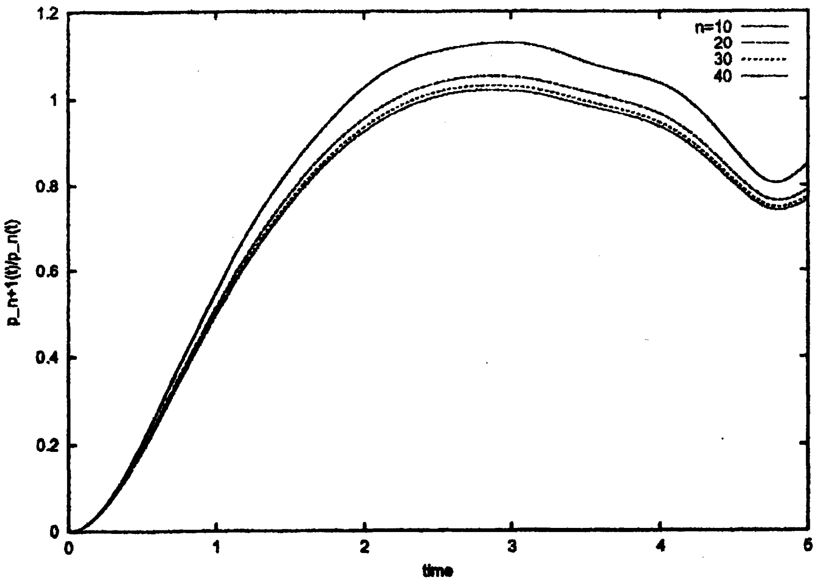

Figure 2: Ratio function $\frac{p_{n}^{(1.5)}}{p_{n+1}^{(1.5)}}$ for $n = 10$, 20, 30 and 40 from $t = 0$ to 5. They converges uniformly to a function rapidly as $n \to \infty$. The integration path is taken from the origin in the positive imaginary time direction.

numerically. Figure 2 shows the ratio function $\frac{p_{n+1}^{(\alpha)}}{p_{n}^{(\alpha)}}$ for $n = 10$, 20, 30 and 40 from $t = 0$ to 5, when the integration path is along the imaginary time axis. It indicates that the ratio function converges uniformly to a function. Since the convergence is quite rapid, we use the ration function for $n = 50$ instead of $g(t)$ and plot its contour plot for various ν in Figure 3. The distribution of the singularity time is topologically the same as the distribution for $\alpha = 1$, which corresponds to the similarity in the analytic properties for $\alpha \geq 1$ in Theorem 4 and 5 for small viscosity and Theorem 14 for large viscosity.

On the other hand, Figure 4 shows the distribution of the singularity times for $\alpha = -0.5$. The singularity times forms a closed curve for each ν. Hence, the distribution has intersections with the positive real-time axis, which indicates that the solution always blows up in finite time. This supports Theorem 16.

5 Conclusion

First, I review the analytic properties of the generalized Constantin-Lax-Majda equation that obtained in the papers[5, 6]. For small viscosity, L^2 norm of the solution diverges in finite time regardless of the order of the diffusion term. For large viscosity, it exists globally when $\alpha \geq 1$ but it always blows up when $\alpha < 0$.

Next, I compute the distribution of critical times of the generalized CLM equation in the complex-time domain. The continuous change of the distribution explains the discontinuity of the singularity formation for $\alpha \geq 1$. Thus the topology of the distribution supports the analytic properties of the equation.

Finally, I compare the generalized CLM equation with the 3-D Navier-Stokes equations. In 3-D Navier-Stokes equations, the solution exists globally in time regardless of the viscosity coefficient when $\alpha > \frac{5}{2}$. Hence, the CLM equation with the generalized viscosity term fails to catch this analytic property of the 3-D Navier-Stokes equations.

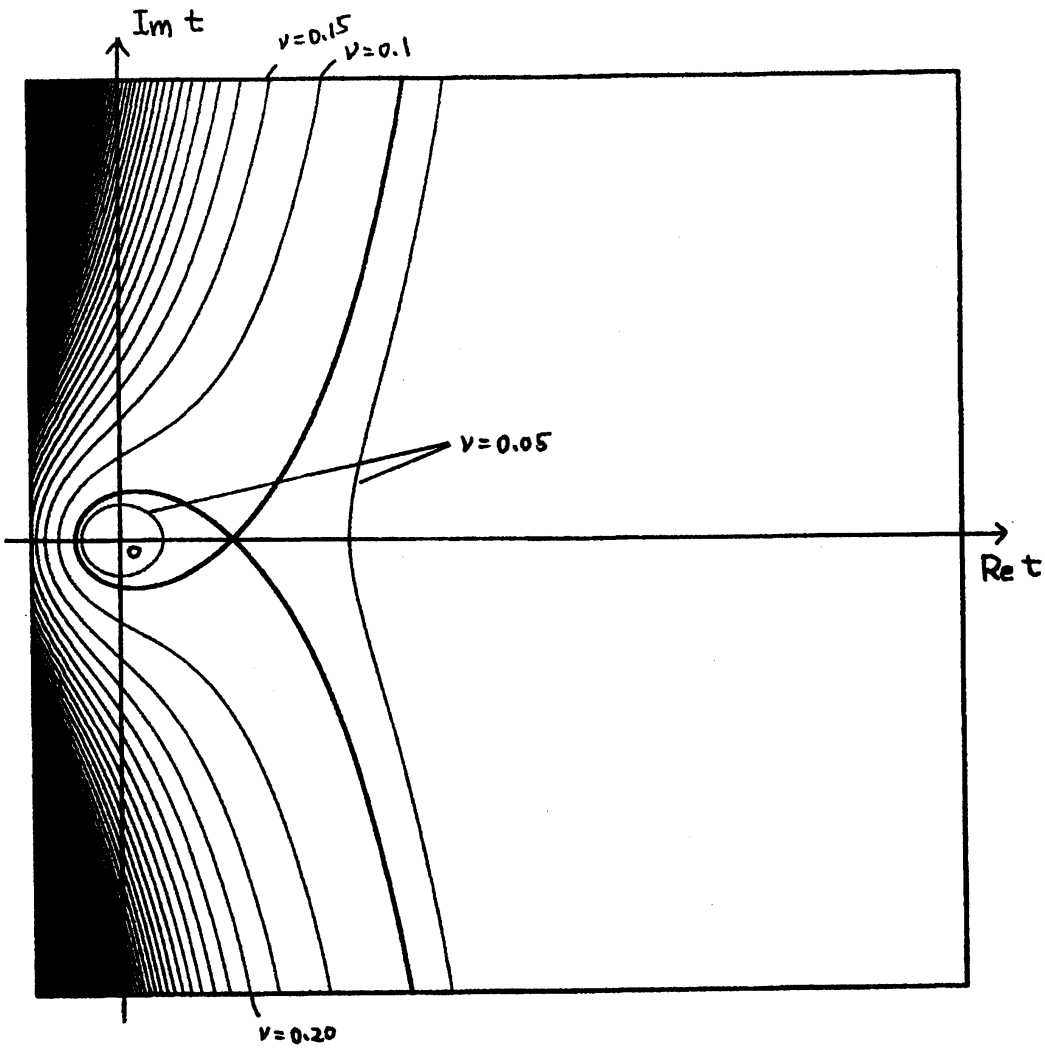

Figure 3: Distribution of singularity time in the complex-time plane for $\alpha = 1.5$. The distribution is topologically the same as that for $\alpha = 1$.

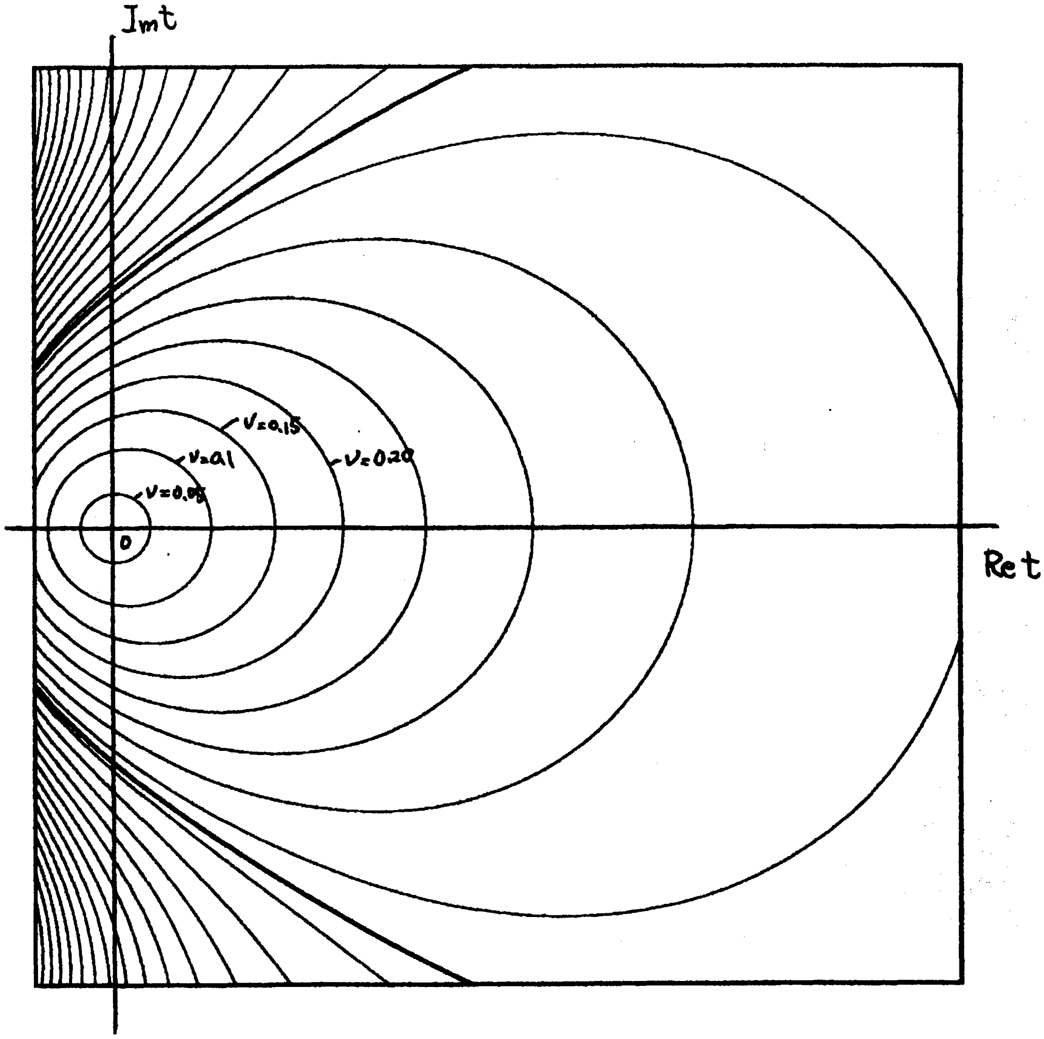

Figure 4: Distribution of singularity time in the complex-time plane for $\alpha = -0.5$.

Acknowledgments

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